(4) $D_4=\begin{vmatrix} 0 & a & b & c \\ a & 0 & c & b \\ b & c & 0 & a \\ c & b & a & 0 \end{vmatrix} \xlongequal[i=2,3,4]{c_1+c_i} (a+b+c)\begin{vmatrix} 1 & a & b & c \\ 1 & 0 & c & b \\ 1 & c & 0 & a \\ 1 & b & a & 0 \end{vmatrix} \xlongequal[i=2,3,4]{r_i+(-1)r_1} (a+b+c)\begin{vmatrix} 1 & a & b & c \\ 0 & -a & c-b & b-c \\ 0 & c-a & -b & a-c \\ 0 & b-a & a-b & -c \end{vmatrix}$

$=(a+b+c)\begin{vmatrix} -a & c-b & b-c \\ c-a & -b & a-c \\ b-a & a-b & -c \end{vmatrix} \xlongequal[c_2+c_3]{c_1+c_3} (a+b+c)\begin{vmatrix} b-a-c & 0 & b-c \\ 0 & a-b-c & a-c \\ b-a-c & a-b-c & -c \end{vmatrix}$

$=(a+b+c)(b-a-c)(a-b-c)\begin{vmatrix} 1 & 0 & b-c \\ 0 & 1 & a-c \\ 1 & 1 & -c \end{vmatrix}$

$\xlongequal{r_3+(-1)r_1} (a+b+c)(b-a-c)(a-b-c)\begin{vmatrix} 1 & 0 & b-c \\ 0 & 1 & a-c \\ 0 & 1 & -b \end{vmatrix}$

$=(a+b+c)(b-a-c)(a-b-c)(c-a-b)$ #.

2.(3). 证明:

$$D_n=\begin{vmatrix} \cos\theta & 1 & 0 & \cdots & 0 & 0 \\ 1 & 2\cos\theta & 1 & \cdots & 0 & 0 \\ 0 & 1 & 2\cos\theta & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 2\cos\theta & 1 \\ 0 & 0 & 0 & \cdots & 1 & 2\cos\theta \end{vmatrix}=\cos n\theta.$$

证明: 当 $n=1$时, $D_1=|\cos\theta|=\cos\theta$, 等式成立.

当 $n=2$时, $D_2=\begin{vmatrix} \cos\theta & 1 \\ 1 & 2\cos\theta \end{vmatrix}=2\cos^2\theta-1=\cos 2\theta$, 等式成立。

下面证 $n>2$, 且假设小于n阶时等式成立. 则 D_n 按最后一行展开, 则

$$D_n=2\cos\theta D_{n-1}-\begin{vmatrix} \cos\theta & 1 & 0 & \cdots & 0 & 0 \\ 1 & 2\cos\theta & 1 & \cdots & 0 & 0 \\ 0 & 1 & 2\cos\theta & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 2\cos\theta & 0 \\ 0 & 0 & 0 & \cdots & 1 & 1 \end{vmatrix}=2\cos\theta D_{n-1}-D_{n-2}$$

即: $D_n = 2\cos\theta D_{n-1} - D_{n-2}$

应用归纳假设. $D_{n-1} = \cos(n-1)\theta$.

$D_{n-2} = \cos(n-2)\theta$

$$\begin{aligned}
\therefore D_n &= 2\cos\theta \cdot \cos(n-1)\theta - \cos(n-2)\theta \\
&= 2\cos\theta \cdot \cos(n-1)\theta - \cos[(n-1)\theta - \theta] \\
&= 2\cos\theta \cdot \cos(n-1)\theta - [\cos(n-1)\theta\cos\theta + \sin(n-1)\theta\sin\theta] \\
&= \cos\theta \cdot \cos(n-1)\theta - \sin(n-1)\theta \cdot \sin\theta \\
&= \cos[\theta + (n-1)\theta] \\
&= \cos n\theta.
\end{aligned}$$

由数学归纳法知, 等式恒成立. #.

线

题

答

作业 P71 1. (1)(3)(5)(7)

计算行列式的值.

P71

1.(1)

$$D_4=\begin{vmatrix}0&1&1&1\\1&0&1&1\\1&1&0&1\\1&1&1&0\end{vmatrix}\xlongequal[i=2.3.4]{c_1+c_i}\begin{vmatrix}3&1&1&1\\3&0&1&1\\3&1&0&1\\3&1&1&0\end{vmatrix}\xlongequal[\substack{r_i+(-1)r_1\\ i=2.3.4}]{\text{第1列提3, 然后}}3\begin{vmatrix}1&1&1&1\\0&-1&0&0\\0&0&-1&0\\0&0&0&-1\end{vmatrix}=-3.$$

(3)

$$D_4=\begin{vmatrix}5&0&4&2\\1&-1&2&1\\4&1&2&0\\1&1&1&1\end{vmatrix}\xlongequal{r_1\leftrightarrow r_4}-\begin{vmatrix}1&1&1&1\\1&-1&2&1\\4&1&2&0\\5&0&4&2\end{vmatrix}\xlongequal[\substack{r_3+(-4)r_1\\ r_4+(-5)r_1}]{r_2+(-1)r_1}-\begin{vmatrix}1&1&1&1\\0&-2&1&0\\0&-3&-2&-4\\0&-5&-1&-3\end{vmatrix}=-\begin{vmatrix}-2&1&0\\-3&-2&-4\\-5&-1&-3\end{vmatrix}$$

$$=-\begin{vmatrix}-2&1&0\\3&2&4\\5&1&3\end{vmatrix}\xlongequal[r_3+(-1)r_1]{r_2+(-2)r_1}-\begin{vmatrix}-2&1&0\\7&0&4\\7&0&3\end{vmatrix}=(-1)\cdot(-1)^{1+2}\begin{vmatrix}7&4\\7&3\end{vmatrix}=-7.$$

(5)

$$D_n=\begin{vmatrix}1&2&2&\cdots&2\\2&2&2&\cdots&2\\2&2&3&\cdots&2\\\cdots&\cdots&\cdots&\cdots&\cdots\\2&2&2&\cdots&n\end{vmatrix}\xlongequal[\substack{\cdots\cdots\\ r_n+(-1)r_1}]{\substack{r_2+(-1)r_1\\ r_3+(-1)r_1}}\begin{vmatrix}1&2&2&\cdots&2\\1&0&0&\cdots&0\\1&0&1&\cdots&0\\\cdots&\cdots&\cdots&\cdots&\cdots\\-1&0&0&\cdots&n-2\end{vmatrix}$$

下一步:按 a_{21} 展开.

$$=1\cdot(-1)^{2+1}\cdot\begin{vmatrix}2&2&\cdots&2\\0&1&\cdots&0\\\cdots&\cdots&\cdots&\cdots\\0&0&\cdots&n-2\end{vmatrix}=(-1)\cdot2\cdot1\cdot2\cdots(n-2)=-2(n-2)!$$

(7)

$$D_n=\begin{vmatrix}1&2&3&\cdots&n-1&n\\1&-1&0&\cdots&0&0\\0&2&-2&\cdots&0&0\\\cdots&\cdots&\cdots&\cdots&\cdots&\cdots\\0&0&0&\cdots&n-1&-(n-1)\end{vmatrix}\xlongequal[i=2,3,\cdots n]{c_1+c_i}\begin{vmatrix}\frac{n(n+1)}{2}&2&3&\cdots&n-1&n\\0&-1&0&\cdots&0&0\\0&2&-2&\cdots&0&0\\\cdots&\cdots&\cdots&\cdots&\cdots&\cdots\\0&0&0&\cdots&n-1&-(n-1)\end{vmatrix}$$

下一步:对第1列提公因子.

$$=\frac{n(n+1)}{2}\begin{vmatrix}1&2&3&\cdots&n-1&n\\0&-1&0&\cdots&0&0\\0&2&-2&\cdots&0&0\\\cdots&\cdots&\cdots&\cdots&\cdots&\cdots\\0&0&0&\cdots&n-1&-(n-1)\end{vmatrix}\xlongequal{\text{按}c_1\text{展开}}\frac{n(n+1)}{2}\begin{vmatrix}-1&0&\cdots&0&0\\2&-2&\cdots&0&0\\\cdots&\cdots&\cdots&\cdots&\cdots\\0&0&\cdots&n-1&-(n-1)\end{vmatrix}=\frac{n(n+1)}{2}\cdot(-1)^{n-1}\cdot(n-1)!$$

$$=\frac{(-1)^{n-1}}{2}\cdot(n+1)!$$

作业 P_{72} 3.

3.

(1)

$$D_n = \begin{vmatrix} a_1+\lambda_1 & a_2 & \cdots & a_n \\ a_1 & a_2+\lambda_2 & \cdots & a_n \\ \cdots & \cdots & \cdots & \cdots \\ a_1 & a_2 & \cdots & a_n+\lambda_n \end{vmatrix} \xlongequal[i=2,3,\cdots,n.]{r_i+(-1)r_1} \begin{vmatrix} a_1+\lambda_1 & a_2 & \cdots & a_n \\ -\lambda_1 & \lambda_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ -\lambda_1 & 0 & \cdots & \lambda_n \end{vmatrix}$$

$$\xlongequal{c_1+\frac{\lambda_1}{\lambda_2}c_2} \begin{vmatrix} a_1+\lambda_1+\frac{\lambda_1}{\lambda_2}a_2 & a_2 & \cdots & a_n \\ 0 & \lambda_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ -\lambda_1 & 0 & \cdots & \lambda_n \end{vmatrix}$$

$$\xlongequal[i=3,4,\cdots n]{c_1+\frac{\lambda_1}{\lambda_i}c_i} \begin{vmatrix} a_1+\lambda_1+\frac{\lambda_1}{\lambda_2}a_2+\frac{\lambda_1}{\lambda_3}a_3+\cdots+\frac{\lambda_1}{\lambda_n}a_n & a_2 & \cdots & a_n \\ 0 & \lambda_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \lambda_n \end{vmatrix}$$

$$= \left(a_1+\lambda_1+\frac{\lambda_1}{\lambda_2}a_2+\frac{\lambda_1}{\lambda_3}a_3+\cdots+\frac{\lambda_1}{\lambda_n}a_n\right)\cdot\lambda_2\cdot\lambda_3\cdots\cdot\lambda_n$$

$$= \left(a_1+\lambda_1+\lambda_1\cdot\sum_{i=2}^{n}\frac{a_i}{\lambda_i}\right)\cdot\prod_{i=2}^{n}\lambda_i$$

$$= \left(\frac{\lambda_1}{\lambda_1}a_1+\lambda_1+\lambda_1\cdot\sum_{i=2}^{n}\frac{a_i}{\lambda_i}\right)\prod_{i=2}^{n}\lambda_i$$

$$= \left(\lambda_1\cdot\sum_{i=1}^{n}\frac{a_i}{\lambda_i}+\lambda_1\right)\prod_{i=2}^{n}\lambda_i$$

$$= \left(1+\sum_{i=1}^{n}\frac{a_i}{\lambda_i}\right)\cdot\lambda_1\cdot\prod_{i=2}^{n}\lambda_i$$

$$= \left(1+\sum_{i=1}^{n}\frac{a_i}{\lambda_i}\right)\cdot\prod_{i=1}^{n}\lambda_i$$

#.

P_{72}

3.(2)

$$D_{n+1}=\begin{vmatrix} a_1^n & a_1^{n-1}b_1 & \cdots & a_1b_1^{n-1} & b_1^n \\ a_2^n & a_2^{n-1}b_2 & \cdots & a_2b_2^{n-1} & b_2^n \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_n^n & a_n^{n-1}b_n & \cdots & a_nb_n^{n-1} & b_n^n \\ a_{n+1}^n & a_{n+1}^{n-1}b_{n+1} & \cdots & a_{n+1}b_{n+1}^{n-1} & b_{n+1}^n \end{vmatrix}$$

观察：

第一行是以 a_1^n 为首项，$\frac{b_1}{a_1}$ 为公比的等比数列，

第二行是以 a_2^n 为首项，b_2/a_2 为公比的等比数列，

……

第 $n+1$ 行是以 a_{n+1}^n 为首项，$\frac{b_{n+1}}{a_{n+1}}$ 为公比的等比数列，

从而每一行提公因式：$a_1^n, a_2^n, \cdots, a_{n+1}^n$，则

$$= a_1^n a_2^n \cdots a_{n+1}^n \cdot \begin{vmatrix} 1 & \frac{b_1}{a_1} & \cdots & \left(\frac{b_1}{a_1}\right)^{n-1} & \left(\frac{b_1}{a_1}\right)^n \\ 1 & \frac{b_2}{a_2} & \cdots & \left(\frac{b_2}{a_2}\right)^{n-1} & \left(\frac{b_2}{a_2}\right)^n \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & \frac{b_n}{a_n} & \cdots & \left(\frac{b_n}{a_n}\right)^{n-1} & \left(\frac{b_n}{a_n}\right)^n \\ 1 & \frac{b_{n+1}}{a_{n+1}} & \cdots & \left(\frac{b_{n+1}}{a_{n+1}}\right)^{n-1} & \left(\frac{b_{n+1}}{a_{n+1}}\right)^n \end{vmatrix}$$

（此行列式为范德蒙行列式（转置））.

$$= a_1^n a_2^n \cdots a_{n+1}^n \cdot \left(\frac{b_2}{a_2}-\frac{b_1}{a_1}\right)\cdot\left(\frac{b_3}{a_3}-\frac{b_1}{a_1}\right)\cdots\cdots\left(\frac{b_{n+1}}{a_{n+1}}-\frac{b_1}{a_1}\right)\cdots$$

$$\cdot\left(\frac{b_3}{a_3}-\frac{b_2}{a_2}\right)\cdots\cdots\left(\frac{b_{n+1}}{a_{n+1}}-\frac{b_2}{a_2}\right)\cdots$$

$$\cdots\left(\frac{b_{n+1}}{a_{n+1}}-\frac{b_n}{a_n}\right).$$

$$=\prod_{i=1}^{n+1} a_i^n \cdot \prod_{1\le j<i\le n}\left(\frac{b_i}{a_i}-\frac{b_j}{a_j}\right).$$

8.

(1)
$$D_n=\begin{vmatrix} a & 0 & \cdots & 1 \\ 0 & a & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 1 & 0 & \cdots & a \end{vmatrix} \xlongequal{r_n+r_1} \begin{vmatrix} a & 0 & \cdots & 1 \\ 0 & a & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ a+1 & 0 & \cdots & a+1 \end{vmatrix} \xlongequal{c_1+(-1)c_n} \begin{vmatrix} a-1 & 0 & \cdots & 1 \\ 0 & a & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a+1 \end{vmatrix}$$

$$=(a-1)(a+1)\cdot a^{n-2}$$

$$=a^{n-2}(a^2-1) \quad \#.$$

(2) 略，显然，课堂举例讲过。

(3)
$$D_{n+1}=\begin{vmatrix} a^n & (a-1)^n & \cdots & (a-n)^n \\ a^{n-1} & (a-1)^{n-1} & \cdots & (a-n)^{n-1} \\ \cdots & \cdots & \cdots & \cdots \\ a^2 & (a-1)^2 & \cdots & (a-n)^2 \\ a & a-1 & \cdots & (a-n) \\ 1 & 1 & \cdots & 1 \end{vmatrix}$$

从最后一行开始；
将最后一行依次与前面的行依次交换。
……
则最后一行成了第一行，第一行成了最后一行，
交换了 $n, n-1, \cdots, 2, 1$，共 $\frac{n(n+1)}{2}$ 次

$$=(-1)^{\frac{n(n+1)}{2}}\cdot\begin{vmatrix} 1 & 1 & \cdots & 1 \\ a & a-1 & \cdots & a-n \\ a^2 & (a-1)^2 & \cdots & (a-n)^2 \\ \cdots & \cdots & \cdots & \cdots \\ a^{n-1} & (a-1)^{n-1} & \cdots & (a-n)^{n-1} \\ a^n & (a-1)^n & \cdots & (a-n)^n \end{vmatrix}$$

类似，从最后一列开始；
将最后一列依次与前面的列依次交换，
……
则最后一列成了第一列，第一列成了最后一列，
交换了 $n, n-1, \cdots n, 1$，共 $\frac{n(n+1)}{2}$ 次。

$$=(-1)^{\frac{n(n+1)}{2}}\cdot(-1)^{\frac{n(n+1)}{2}}\begin{vmatrix} 1 & 1 & \cdots & 1 & 1 \\ a-n & a-(n-1) & \cdots & a-1 & a \\ (a-n)^2 & (a-(n-1))^2 & \cdots & (a-1)^2 & a^2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ (a-n)^{n-1} & (a-(n-1))^{n-1} & \cdots & (a-1)^{n-1} & a^{n-1} \\ (a-n)^n & (a-(n-1))^{n-1} & \cdots & (a-1)^n & a^n \end{vmatrix}$$

此为 $n+1$ 阶 Vandermonde 行列式。
中间的一般项为：$a-(n-i)$，$a-(n-j)$

$$=\prod_{1\le j<i\le n+1}\left[(a-(n-i))-(a-(n-j))\right]$$

$$=\prod_{1\le j<i\le n+1}(i-j) \quad \#.$$

答 题 线

(4) $D_{2n} = \begin{vmatrix} a_n & & & & & b_n \\ & \ddots & & & \iddots & \\ & & a_1 & b_1 & & \\ & & c_1 & d_1 & & \\ & \iddots & & & \ddots & \\ c_n & & & & & d_n \end{vmatrix}$ 此题与课堂讲解题类似..

$$\xlongequal[c_2 \leftrightarrow c_{2n}]{r_2 \leftrightarrow r_{2n}} \left|\begin{array}{cc:c} a_n & b_n & \\ c_n & d_n & 0 \\ \hdashline 0 & & D_{2(n-1)} \end{array}\right| = (a_n d_n - b_n c_n) \cdot D_{2(n-1)}$$

即: $D_{2n} = (a_n d_n - b_n c_n) D_{2(n-1)}$ ，利用递推关系.

$$D_{2(n-1)} = (a_{n-1} d_{n-1} - b_{n-1} c_{n-1}) D_{2(n-2)}$$

$$\cdots\cdots$$

$$D_2 = \begin{vmatrix} a_1 & b_1 \\ c_1 & d_1 \end{vmatrix} = (a_1 d_1 - b_1 c_1).$$

$$\therefore D_{2n} = (a_n d_n - b_n c_n)(a_{n-1} d_{n-1} - b_{n-1} c_{n-1}) \cdots (a_1 d_1 - b_1 c_1)$$

$$= \prod_{i=1}^{n} (a_i d_i - b_i c_i) \ \#.$$

(5). $D_n = \begin{vmatrix} 1+a_1 & a_1 & \cdots & a_1 \\ a_2 & 1+a_2 & \cdots & a_2 \\ \cdots & \cdots & \cdots & \cdots \\ a_n & a_n & \cdots & 1+a_n \end{vmatrix} \xlongequal[i=2,\cdots n]{r_1 + r_i} \begin{vmatrix} 1+\sum_{i=1}^{n} a_i & 1+\sum_{i=1}^{n} a_i & \cdots & 1+\sum_{i=1}^{n} a_i \\ a_2 & 1+a_2 & \cdots & a_2 \\ \cdots & \cdots & & \cdots \\ a_n & a_n & & 1+a_n \end{vmatrix}$

$$= \left(1+\sum_{i=1}^{n} a_i\right) \begin{vmatrix} 1 & 1 & \cdots & 1 \\ a_2 & 1+a_2 & \cdots & a_2 \\ \cdots & \cdots & \cdots & \cdots \\ a_n & a_n & \cdots & 1+a_n \end{vmatrix} \xlongequal[i=2,3,\cdots n]{c_i + (-1) c_1} \left(1+\sum_{i=1}^{n} a_i\right) \begin{vmatrix} 1 & 0 & \cdots & 0 \\ a_2 & 1 & \cdots & 0 \\ \vdots & \vdots & \cdots & \cdots \\ a_2 & 0 & \cdots & 1 \end{vmatrix}$$

$$= 1 + \sum_{i=1}^{n} a_i$$

$$= 1 + a_1 + a_2 + \cdots + a_n$$

(6) $D_n=\det(a_{ij})$, $a_{ij}=|i-j|$

$$
=\begin{vmatrix} 0 & 1 & 2 & \cdots & n-1 \\ 1 & 0 & 1 & \cdots & n-2 \\ 2 & 1 & 0 & \cdots & n-3 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ n-1 & n-2 & n-3 & \cdots & 0 \end{vmatrix}
\xlongequal[\substack{r_{n-1}+(-1)r_{n-2} \\ \cdots\cdots \\ r_2+(-1)r_1}]{r_n+(-1)r_{n-1}}
\begin{vmatrix} 0 & 1 & \cdots & n-2 & n-1 \\ 1 & -1 & \cdots & -1 & -1 \\ 1 & 1 & \cdots & -1 & -1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & \cdots & 1 & -1 \end{vmatrix}
$$

$$
\xlongequal[i=1,2,\cdots,n-1]{c_i+c_n}
\begin{vmatrix} n-1 & n & \cdots & 2n-3 & n-1 \\ 0 & -2 & \cdots & -2 & -1 \\ 0 & 0 & \cdots & -2 & -1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & -1 \end{vmatrix}
=(-1)^{n-1}\cdot(n-1)\cdot 2^{n-2}
$$

#.

(7)
$$
\begin{vmatrix} 1+a_1 & 1 & \cdots & 1 \\ 1 & 1+a_2 & \cdots & 1 \\ \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & \cdots & 1+a_n \end{vmatrix}
$$

$$
\xlongequal[i=2,3,\cdots,n]{r_i+(-1)r_1}
\begin{vmatrix} 1+a_1 & 1 & \cdots & 1 \\ -a_1 & a_2 & \cdots & 0 \\ \vdots & & \ddots & \\ -a_1 & 0 & & a_n \end{vmatrix}
\xlongequal{c_1+\frac{a_1}{a_2}c_2}
\begin{vmatrix} 1+a_1+\frac{a_1}{a_2} & 1 & \cdots & 1 \\ 0 & a_2 & & 0 \\ \cdots & \cdots & \ddots & \\ -a_1 & 0 & & a_n \end{vmatrix}
$$

$$
\xlongequal[i=3,\cdots,n]{c_1+\frac{a_1}{a_i}c_i}
\begin{vmatrix} 1+a_1+\frac{a_1}{a_2}+\cdots+\frac{a_1}{a_n} & 1 & \cdots & 1 \\ 0 & a_2 & \cdots & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & a_n \end{vmatrix}
$$

$$
=(a_1+\frac{a_1}{a_1}+\frac{a_1}{a_2}+\cdots+\frac{a_1}{a_n})\cdot a_2\cdots a_n
$$

$$
=a_1(1+\sum_{i=1}^{n}\frac{1}{a_i})\cdot a_2\cdots a_n
$$

$$
=a_1a_2\cdots a_n(1+\sum_{i=1}^{n}\frac{1}{a_i})
$$

#.